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Color degree and heterochromatic cycles in edge-colored graphs☆

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ABSTRACT

Given a graph G and an edge-coloring C of G, a heterochromatic cycle of G is a cycle in which any pair of edges have distinct colors. Let $d^c(v)$, named the color degree of a vertex v, be defined as the maximum number of edges incident with v that have distinct colors. In this paper, some color degree conditions for the existence of heterochromatic cycles are obtained.

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1. Introduction

We use [4] for terminology and notation not defined here. Let $G = (V, E)$ be a graph. An *edge-coloring* of G is a function $C : E \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. If G is assigned such a coloring C, then we say that G is an *edge-colored graph*. Let $C(e)$ be the *color* of the edge $e \in E$. For a vertex v and a subset $S \subseteq V(G)$, let $C(v, S)$ denote the set $\{C(vu) \mid u \in S\}$ and $CN(v)$ denote the set $\{C(e) \mid e$ is incident with $v\}$. For a subgraph H of G, let $C(H) = \{C(e) \mid e \in E(H)\}$ and $c(H) = |C(H)|$.

A subgraph H of G is called *heterochromatic*, or *rainbow*, or *multicolored* if any pair of edges in H have distinct colors. The heterochromatic subgraphs have received increasing attention recently as mentioned below.

Suzuki [16] gave a necessary and sufficient condition for the existence of a heterochromatic spanning tree in an edge-colored connected graph. Shor [15] proved that every $n \times n$ Latin square has a partial transversal of length at least $n - 5.53(\log_2 n)^2$, namely every properly edge-colored complete bipartite graph $K_{n,n}$ with n colors has a heterochromatic matching with at least $n - 5.53(\log_2 n)^2$ edges. Li and Wang [12,13] studied the heterochromatic matchings in edge-colored bipartite graphs. It can

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be easily seen that the heterochromatic matchings in edge-colored bipartite graphs are, in another terminology, matchings in tripartite 3-uniform hypergraphs.

Chen and Li [7] studied the long heterochromatic path under the color degree conditions (see the definition in Section 2). Albert et al. [1] showed that if n is sufficiently large and the edges of the complete graph K_n are colored so that no color appears more than $\lceil cn \rceil$ times, where $c < 1/32$ is a constant, then there is a heterochromatic Hamiltonian cycle. For more references, see [2,8–11,17].

For integer l ($l \geq 3$), let HC_l denote a heterochromatic cycle of length l. The existence of heterochromatic cycles has been studied in [5] by Broersma et al. and they obtained the following results.

Theorem 1.1 ([5]). *Let G be an edge-colored graph of order n such that $c(G) \geq n$. Then G contains a heterochromatic cycle of length at least $\frac{2c(G)}{n-1}$.*

Theorem 1.2 ([5]). *Let G be an edge-colored graph of order $n \geq 4$ such that $|CN(u) \cup CN(v)| \geq n - 1$ for every pair of vertices u and v of G; then G contains at least one HC_3 or one HC_4.*

2. The main results

First, we give some definitions. For a vertex $v \in V(G)$, a *color neighbourhood* of v is defined as a set $T \subseteq N(v)$ such that the colors of the edges between v and T are pairwise distinct. A *maximum color neighborhood* $N^c(v)$ of v is a color neighborhood of v with maximum size. Let $d^c(v) = |N^c(v)|$; we call it the *color degree* of v. Clearly $d^c(v) = |CN(v)|$.

We are interested in Dirac type conditions (i.e., minimum color degree conditions) for the existence of heterochromatic cycles, in particular the shortest heterochromatic cycles and the longest heterochromatic cycles.

We begin with a study of the existence of a heterochromatic cycle. Existence of a heterochromatic cycle can be insured by Theorem 1.1 when $c(G) \geq n$. Under color degree conditions, we have the following result.

Theorem 2.1. *Let G be an edge-colored graph of order n, $n \geq 3$. If for each vertex v of G, $d^c(v) \geq \frac{n+1}{2}$, then G has a heterochromatic cycle.*

We also get some results for the existence of HC_3 or HC_4 under some color degree conditions.

Theorem 2.2. *Let G be an edge-colored graph of order n, $n \geq 3$. If for each vertex v of G, $d^c(v) \geq (\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}$, then G has either an HC_3 or an HC_4.*

Note that $\frac{4\sqrt{7}}{7} - 1 = 0.515\ldots$ and $3 - \frac{4\sqrt{7}}{7} = 1.488\ldots$.

Theorem 2.3. *Let G be an edge-colored graph of order n, $n \geq 3$. If for each vertex v of G, $d^c(v) \geq \frac{\sqrt{7}+1}{6}n$, then G has an HC_3.*

Note that $\frac{\sqrt{7}+1}{6} = 0.608\cdots$. We believe that there is room for improvement in the above bound, and we propose the following conjecture.

Conjecture 2.4. *Let G be an edge-colored graph of order n, $n \geq 3$. If for each vertex v of G, $d^c(v) \geq \frac{n+1}{2}$, then G has an HC_3.*

The following example shows that if the above conjecture is true, it would be best possible. For any even integer n, let $B_{n/2,n/2}$ be an edge-proper-colored complete balanced bipartite graph of order n. For every vertex v of $B_{n/2,n/2}$, it holds that $d^c(v) = \frac{n}{2}$, and $B_{n/2,n/2}$ has no HC_3.

For the existence of a heterochromatic cycle, it is natural to consider the following problem: determining a function $f(n)$ as small as possible such that for any edge-colored graph G of order n, if for each vertex v of G, $d^c(v) \geq f(n)$, then G contains a heterochromatic cycle.

The following two propositions show that $f(n)$ must be greater than $\log_2 n$.

Proposition 2.5. *For any positive integer k, there exists an edge-colored bipartite graph B of order $n = 2^k$ such that for each vertex v of B, $d^c(v) = k = \log_2 n$, and B has no heterochromatic cycle.*

To show Proposition 2.5, we construct an example by induction. Let G_1 be an edge e with color $C(e) = 1$. Given graph G_i for $i \geq 1$, we define G_{i+1} as follows. First we construct a graph G_i' which is a copy of G_i. For each vertex $u \in V(G_i)$, join u and u', in which u' is the copy of u in G_i', then color the new edge uu' with color $i+1$. The new edge-colored graph is denoted by G_{i+1}.

Now $B = G_i$ is an edge-colored bipartite graphs of order $n = 2^i$. For every vertex v, it holds that $d^c(v) = i = \log_2 n$. Clearly B has no heterochromatic cycle.

Proposition 2.6. *For any positive integer k, there exists an edge-colored complete graph K of order $n = 2^k$ such that for each vertex v of K, $d^c(v) = k = \log_2 n$, and K has no heterochromatic cycle.*

We construct graphs in a way slightly differing from that in the above example. Let G_1^* be an edge e with colors $C(e) = 1$. Given graph G_i^* for $i \geq 1$, we construct G_{i+1}^* as follows. Let the graph G_i^{**} be a copy of G_i^*. For any $u \in V(G_i^*)$, $u' \in V(G_i^{**})$, join u and u' and let $C(uu') = i+1$. The new edge-colored graph is denoted by G_{i+1}^*.

Now $K = G_i^*$ is an edge-colored complete graph of order $n = 2^i$. It holds that $d^c(v) = i = \log_2 n$, for every vertex v of K. Clearly, K has no heterochromatic cycle.

For long heterochromatic cycles, we show the following result.

Theorem 2.7. *Let G be an edge-colored graph of order n, $n \geq 8$. If for each vertex v of G, $d^c(v) \geq d \geq \frac{3n}{4} + 1$, then G has an HC_l such that $l \geq d - \frac{3n}{4} + 2$.*

Since the proof of Theorem 2.1 is analogous to that of Theorem 2.3 and simpler than it, we omit this proof. The proofs of Theorems 2.2, 2.3 and 2.7 will be given in Section 3.

3. Proofs of the main results

First, we give some preliminaries. Caccetta and Häggkvist [6] proposed the following conjecture.

Conjecture 3.1 ([6]). *Any digraph D on m vertices with minimum outdegree at least r contains a directed cycle of length at most $\lceil \frac{m}{r} \rceil$.*

A particularly interesting special case that is still open: any digraph on m vertices with minimum outdegree at least $\frac{m}{3}$ contains a directed triangle. Short of proving this, one may seek a value α as small as possible such that every digraph on m vertices with minimum outdegree at least αm contains a directed triangle. Cacdetta and Häggkvist showed that $\alpha \leq \frac{3-\sqrt{5}}{2} = 0.3819\cdots$. Bondy [3] proved that $\alpha \leq \frac{2\sqrt{6}-3}{5} = 0.3797\cdots$. This result was improved by Shen [14] as follows.

Lemma 3.2 ([14]). *If $\alpha = 3 - \sqrt{7} = 0.3542\cdots$, then any digraph on m vertices with minimum outdegree at least αm contains a directed triangle.*

The following result is clear and will also be used in our proof.

Lemma 3.3. *Every simple digraph on m-vertices with minimum outdegree at least 1 has a directed cycle.*

Proof of Theorem 2.3. If $n = 3, 4$, clearly Theorem 2.3 holds. So we assume that $n \geq 5$. By contradiction, suppose G is an edge-colored graph with $d^c(v) \geq \frac{\sqrt{7}+1}{6}n$ for every vertex v of G, and G contains no heterochromatic triangle. Let v be an arbitrary vertex of G. Choose a maximum color neighborhood $N^c(v)$ of v. Assume that $T = N^c(v) = \{v_1, v_2, \ldots, v_k\}$, where $k = d^c(v)$. Since G has no heterochromatic triangle, if $e = v_iv_j \in E(G[T])$, $1 \leq i, j \leq k$, then $C(e) = C(vv_i)$ or $C(e) = C(vv_j)$.

Give an orientation of $G[T]$ using the following rule: for an edge $e = v_iv_j$, if $C(e) = vv_i$, then the orientation of v_iv_j is from v_j to v_i; otherwise the orientation is from v_i to v_j. After the orientation, the oriented graph is denoted by D. For any vertex $u \in V(D)$, let $N_D^+(u)$ denote the outneighbors of u in D and $d_D^+(u) = |N_D^+(u)|$.

Claim 1. There exists a directed cycle in D.

Proof. Otherwise, by Lemma 3.3, we know that there exists a vertex v_j of D such that $d_D^+(v_j) = 0$. Let $N^c(v_j)$ be a maximum color neighborhood of v_j in G. Now we conclude that $|N^c(v_j) \setminus (T \cup \{v\})| \geq d^c(v_j) - 1$. Thus it follows that

$$\begin{aligned} n &\geq |N^c(v_j) \setminus (T \cup \{v\})| + |T| + |v| \geq d^c(v_j) - 1 + d^c(v) + 1 \\ &\geq 2\frac{\sqrt{7}+1}{6}n \geq 2\left(\frac{n+1}{2}\right) = n + 1. \end{aligned}$$

This contradiction completes the proof. □

Claim 2. Let q ($q \geq 3$) be an integer. If $\overrightarrow{C_q}$ is a directed cycle in D, then C_q is a heterochromatic cycle in G.

Proof. Without loss of generality, we assume that D has a directed cycle $\overrightarrow{C_q}: v_1 \to v_2 \to \cdots \to v_q \to v_1$. By the above orientation rule, we conclude that $C(v_iv_{i+1}) = C(vv_{i+1})$ for $1 \leq i \leq q-1$ and $C(v_qv_1) = C(vv_1)$. Since $T = N^c(v)$ is a maximum color neighborhood of v, we have that $C(vv_i) \neq C(vv_j)$ for $i \neq j$. Thus C_q is a heterochromatic cycle in G. □

Since G has no heterochromatic triangle, by Claim 2, D has no directed triangle. By Lemma 3.2, we conclude that there exists a vertex v_i in D such that $d_D^+(v_i) < \alpha V(D) = \alpha d^c(v)$. Let $G_0 = G[T \cup \{v\}]$ and $N_{G_0}^c(v_i)$ denote a maximum color neighborhood of v_i in graph G_0. By the orientation rule, $|N_{G_0}^c(v_i)| = |N_D^+(v_i)| + |v| = |d_D^+(v_i)| + 1 < \alpha d^c(v) + 1$. Let $N^c(v_i)$ be a maximum color neighborhood of v_i in G. It follows that $|N^c(v_i) \setminus (T \cup \{v\})| \geq d^c(v_i) - |N_{G_0}^c(v_i)| > d^c(v_i) - \alpha d^c(v) - 1$. Thus it holds that

$$n \geq |N^c(v_i) \setminus (T \cup \{v\})| + |T| + |v| > d^c(v_i) + (1-\alpha)d^c(v) \geq (2-\alpha)\frac{\sqrt{7}+1}{6}n = n.$$

This contradiction completes the proof of Theorem 2.3. □

Proof of Theorem 2.2. By contradiction, suppose that G is an edge-colored graph such that $d^c(v) \geq (\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}$ for every vertex $v \in V(G)$, and G contains neither HC_3 nor HC_4.

For an edge uv, let $N_1^c(u), N_1^c(v)$ denote the maximum color neighborhoods of u, v, respectively, such that $v \in N_1^c(u)$, $u \in N_1^c(v)$ and $|N_1^c(u) \cup N_1^c(v)|$ is maximum. Let $N^c(u, v)$ denote $N_1^c(u) \cup N_1^c(v)$. Choose an edge $uv \in E(G)$ such that $|N^c(u, v)|$ is maximum.

Assume that $N_1^c(u) = \{v, u_1, u_2, \ldots, u_s\}$ and $N_1^c(v) \setminus N_1^c(u) = \{u, v_1, v_2, \ldots, v_t\}$, in which $s = d^c(u) - 1$. Let $X = \{u_1, \ldots, u_s, v_1, \ldots, v_t\}$. Note that $|N^c(u, v)| = s + t + 2$. We have the following claim.

Claim 1. Suppose $e \in E(G[X])$; then

(i) if $e = u_iu_j$ ($1 \leq i, j \leq s$), then $C(e) \in \{C(uu_i), C(uu_j)\}$;
(ii) if $e = v_iv_j$ ($1 \leq i, j \leq t$), then $C(e) \in \{C(vv_i), C(vv_j)\}$;
(iii) if $e = u_iv_j$ ($1 \leq i \leq s$, $1 \leq j \leq t$) and $C(uu_i) \neq C(vv_j)$, then $C(e) \in \{C(uu_i), C(vv_j), C(uv)\}$.

Proof. Clearly (i) and (ii) hold; otherwise we can obtain an HC_3, which is a contradiction. If (iii) does not hold, then there exists an edge $e = u_iv_j$ ($1 \leq i \leq s, 1 \leq j \leq t$) such that $C(uu_i) \neq C(vv_j)$ and $C(e) \notin \{C(uu_i), C(vv_j), C(uv)\}$. Since $v, u_i \in N_1^c(u)$, $C(uu_i) \neq C(uv)$. Similarly, we obtain that $C(vv_j) \neq C(uv)$. Thus we can get an $HC_4 = uvv_ju_iu$, which is a contradiction. □

Given graph $G[X]$, let D_1 denote the digraph obtained by the following operations.

(1) Remove the edges $e = v_iu_j$ if $C(e) = C(uv)$ or $C(uu_i) = C(vv_j)$, $1 \leq i \leq s$ and $1 \leq j \leq t$.
(2) Orient the rest of the edges by applying the following rule: for an edge xy, if $C(xy) = C(uy)$ or $C(xy) = C(vy)$, then the orientation of xy is from x to y; otherwise, by Claim 1, $C(xy) = C(ux)$ or $C(xy) = C(vx)$, and then the orientation of xy is from y to x.

For any vertex $w \in V(D_1)$, let $N_{D_1}^+(w)$ denote the outneighbors of w in D_1 and $d_{D_1}^+(w) = |N_{D_1}^+(w)|$. Let $G_0 = G[X \cup \{u, v\}]$.

Claim 2. If there exists a directed triangle $\overrightarrow{C_3}$ in D_1, then C_3 is a heterochromatic triangle in G.

Proof. Suppose that $\overrightarrow{C_3} : x \to y \to z \to x$ is a directed triangle in D_1. If $x, y, z \in N_1^c(u)$, then by the orientation rule, it holds that $C(xy) = C(uy)$, $C(yz) = C(uz)$ and $C(zx) = C(ux)$. By the definition of $N_1^c(u)$, we conclude that $C(ux), C(uy), C(uz)$ are distinct pairwise. Thus $C_3 = xyzx$ is a heterochromatic triangle of G. Similarly, if $x, y, z \in N_1^c(v) \setminus N_1^c(u)$, we also conclude that C_3 is a heterochromatic triangle of G.

Thus, without loss of generality, we assume that $x, y \in N_1^c(u)$ and $z \in N_1^c(v) \setminus N_1^c(u)$. By the orientation rule, $C(xy) = C(uy)$, $C(yz) = C(vz)$ and $C(zx) = C(ux)$. By the definition of $N_1^c(u)$ and Claim 1(iii), we have that $C(ux), C(uy)$ and $C(vz)$ are distinct pairwise; then it follows that C_3 is a heterochromatic triangle of G. □

Let $\alpha = 3 - \sqrt{7}$. By Claim 2, there is no directed triangle in D_1. By Lemma 3.2, there is a vertex w such that $d_{D_1}^+(w) < \alpha|V(D_1)| = \alpha(s + t) = \alpha(d^c(u) + t - 1)$. Without loss of generality, assume that $w \in N_1^c(u)$. Let $N_{G_0}^c(w)$ denote a maximum color neighborhood of w in G_0. Note that, in the deleting operation, at most two edges incident with w are deleted; then $|N_{G_0}^c(w)| \le |N_{D_1}^+(w)| + |v|(\text{or } |u|) + 2 = d_{D_1}^+(w) + 3$. Let $N^c(w)$ be a maximum color neighborhood of w in G. It follows that

$$|N^c(w) \setminus (X \cup \{u, v\})| \ge d^c(w) - |N_{G_0}^c(w)| > d^c(w) - \alpha(d^c(u) + t - 1) - 3.$$

If $d^c(w) - \alpha(d^c(u) + t - 1) - 3 > t$, then we consider the edge uw and it follows that

$$\begin{aligned} |N^c(u, w)| &\ge |\{u_1, u_2, \ldots, u_s\} \cup \{v\}| + |N^c(w) \setminus (X \cup \{u, v\})| + |u| \\ &> s + t + 2 \\ &= |N^c(u, v)|, \end{aligned}$$

which is a contradiction with the choice of uv.

Thus $d^c(w) - \alpha(d^c(u) + t - 1) - 3 \le t$; then $t \ge \frac{d^c(w)}{1+\alpha} - \frac{\alpha d^c(u)}{1+\alpha} + \frac{\alpha-3}{1+\alpha}$. It follows that

$$\begin{aligned} n &\ge |X| + |u| + |v| + |N^c(w) \setminus (X \cup \{u, v\})| \\ &> d^c(u) + t - 1 + 2 + d^c(w) - \alpha(d^c(u) + t - 1) - 3 \\ &\ge (1-\alpha)d^c(u) + d^c(w) + (1-\alpha)\left(\frac{d^c(w)}{1+\alpha} - \frac{\alpha d^c(u)}{1+\alpha} + \frac{\alpha-3}{1+\alpha}\right) + \alpha - 2 \\ &\ge \frac{1-\alpha}{1+\alpha}d^c(u) + \frac{2}{1+\alpha}d^c(w) + \frac{3\alpha-5}{1+\alpha}. \end{aligned}$$

Since $d^c(v) \ge (\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}$ for every vertex $v \in V(G)$ and $\alpha = 3 - \sqrt{7}$, the above inequality is

$$n > \frac{3-\alpha}{1+\alpha}\left[\left(\frac{4\sqrt{7}}{7} - 1\right)n + 3 - \frac{4\sqrt{7}}{7}\right] + \frac{3\alpha-5}{1+\alpha} \ge n.$$

This contradiction completes the proof of Theorem 2.2. □

Proof of Theorem 2.7. By contradiction, since $d^c(v) \ge \frac{3n}{4} + 1 > \frac{n+1}{2}$, by Theorem 2.1, G has a heterochromatic cycle. We choose a longest heterochromatic cycle HC_l of length l. If the conclusion fails, it holds that $l < d - \frac{3n}{4} + 2$. Note that now $d > \frac{3n}{4} + 1$.

Assume that $xy \in E(HC_l)$. Let $N^c(x), N^c(y)$ be the maximum color neighborhoods of x, y, respectively. Choose a set S_x such that:

(R_1) $S_x \subseteq N^c(x) \setminus V(HC_l)$;
(R_2) for each $v \in S_x$, $C(xv) \notin C(HC_l)$;
(R_3) subject to (R_1), (R_2), $|S_x|$ is maximum.

Similarly, choose a set S_y such that:

(R'_1) $S_y \subseteq N^c(y) \setminus V(\mathrm{HC}_l)$;

(R'_2) for each $v \in S_y$, $C(yv) \notin C(\mathrm{HC}_l)$;

(R'_3) subject to (R'_1), (R'_2), $|S_y|$ is maximum.

Let $P = S_x \cap S_y$ and $p = |P|$. Now we have the following claims.

Claim 1. $p \geq 2d - n + 6 - 3l > 0$.

Proof. Clearly, we conclude that $|S_x| \geq d^c(x) - l - (l-3) \geq d + 3 - 2l$. Similarly, $|S_y| \geq d + 3 - 2l$. So $p \geq |S_x| + |S_y| - (n-l) \geq 2d - n + 6 - 3l > 0$. □

Claim 2. If $u \in P$, then $C(ux) = C(uy)$.

Proof. Otherwise, if $C(ux) \neq C(uy)$, since $C(ux), C(uy) \notin C(\mathrm{HC}_l)$, we can get a heterochromatic cycle: $\mathrm{HC}_l \cup \{xu, uy\} \setminus \{xy\}$ of length $l+1$, which is a contradiction. □

Claim 3. If $uv \in E(G[P])$, then $C(uv) \in \{C(ux), C(vy), C(\mathrm{HC}_l) \setminus C(xy)\}$.

Proof. If $uv \in E(G[P])$, then by Claim 2, $C(ux) = C(uy)$ and $C(vx) = C(vy)$. Clearly, we have that $C(ux) \neq C(vy)$. So if $C(uv) \notin \{C(ux), C(vy), C(\mathrm{HC}_l) \setminus C(xy)\}$, then we have a heterochromatic cycle: $\mathrm{HC}_l \cup \{ux, uv, vy\} \setminus \{xy\}$ of length $l+2$, which is a contradiction. □

Given graph $G[P]$, let D_2 denote the digraph obtained by the following operations.

(1) Remove the edges uv if $C(uv) \in C(\mathrm{HC}_l) \setminus C(xy)$.

(2) Orient the rest of the edges by applying the following rule: for an edge uv, if $C(uv) = C(ux)$, then the orientation of uv is from v to u; otherwise, by Claim 3, $C(uv) = C(vy)$, and then the orientation of uv is from u to v.

Let v_0 be a vertex in D_2 with minimum outdegree, $d^+_{D_2}(v_0)$. Clearly, $d^+_{D_2}(v_0) \leq \frac{p-1}{2}$. Let $N^c(v_0)$ denote a maximum color neighborhood of v_0 in G. Assume that $N^c(v_0) = V_1 \cup V_2 \cup V_3 \cup V_4$, in which

$$V_1 = \{v \mid v \in P \text{ and } C(v_0v) \notin C(\mathrm{HC}_l)\},$$
$$V_2 = \{v \mid v \in V(\mathrm{HC}_l) \text{ and } C(v_0v) \notin C(\mathrm{HC}_l)\},$$
$$V_3 = \{v \mid v \in P \cup V(\mathrm{HC}_l) \text{ and } C(v_0v) \in C(\mathrm{HC}_l)\},$$
$$V_4 = \{v \mid v \notin P \cup V(\mathrm{HC}_l)\},$$

and $V_i \cap V_j = \phi$, for $1 \leq i \neq j \leq 4$. We can conclude that $|V_1| \leq d^+_{D_2}(v_0) + 1 \leq \frac{p-1}{2} + 1$ and $|V_3| \leq l$.

Claim 4. $|V_1| + |V_2| \leq \frac{p-1}{2} + \frac{l-1}{2}$.

Proof. First, we conclude that $|V_2| \leq \frac{l-1}{2}$. Otherwise if $|V_2| > \frac{l-1}{2}$, by $C(xv_0) = C(yv_0) \notin C(\mathrm{HC}_l)$, then there exist two consecutive vertices v_i, v_{i+1} of HC_l such that $C(v_0v_i), C(v_0v_{i+1}) \notin C(\mathrm{HC}_l)$ and $C(v_0v_i) \neq C(v_0v_{i+1})$. Thus we can get a heterochromatic cycle: $\mathrm{HC}_l \cup \{v_iv_0, v_0v_{i+1}\} \setminus \{v_iv_{i+1}\}$ of length $l+1$, which is a contradiction. So if $|V_1| \leq \frac{p-1}{2}$, then $|V_1| + |V_2| \leq \frac{p-1}{2} + \frac{l-1}{2}$.

Moreover if $|V_1| = \frac{p-1}{2} + 1$, then $C(xv_0) \in C(v_0, V_1)$. By the definition of a maximum color neighborhood $N^c(v_0)$ of v_0 and $V_1 \cap V_2 = \phi$, we conclude that $C(xv_0) \notin C(v_0, V_2)$. If $|V_2| > \frac{l-3}{2}$, using the same method as above, we can get a heterochromatic cycle of length $l+1$, which is a contradiction. So it holds that $|V_2| \leq \frac{l-3}{2}$; then $|V_1| + |V_2| \leq \frac{p-1}{2} + \frac{l-1}{2}$. □

Now we complete the proof of Theorem 2.7 as follows. Since $\sum_{i=1}^{4} |V_i| = d^c(v_0) \geq d$ and $V_i \cap V_j = \phi$, for $1 \leq i \neq j \leq 4$, $|V_4| \geq d - \sum_{i=1}^{3} |V_i| \geq d - l - \frac{p-1}{2} - \frac{l-1}{2}$. Clearly $V_4 \subseteq V(G) \setminus (P \cup V(\mathrm{HC}_l))$. So we have that $d - l - \frac{p-1}{2} - \frac{l-1}{2} \leq n - p - l$. It follows that $p \leq 2(n-d) + l - 2$. By Claim 1, we also have that $p \geq 2d - n + 6 - 3l$. Thus $l \geq d - \frac{3n}{4} + 2$. This contradiction completes the proof. □

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